SMALL VALUES OF THE LUSTERNIK-SCHNIRELMANN CATEGORY FOR MANIFOLDS

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Abstract. We prove that manifolds of Lusternik-Schnirelmann category 2 necessarily have free fundamental group. We thus settle a 1992 conjecture of Gomez-Larrañaga and Gonzalez-Acuña, by generalizing their result in dimension 3, to all higher dimensions. We also obtain some general results on the relations between the fundamental group of a closed manifold M, the dimension of M, and the Lusternik-Schnirelmann category of M, and relate the latter to the systolic category of M.

Contents

1. Introduction

We follow the normalization of the Lusternik-Schnirelmann category (LS category) used in the recent monograph [CLOT03] (see Section 3

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for a definition). Spaces of LS category 0 are contractible, while a closed manifold of LS category 1 is homotopy equivalent (and hence homeomorphic) to a sphere.

The characterization of closed manifolds of LS category 2 was initiated in 1992 by J. Gomez-Larrañaga and F. Gonzalez-Acuña [GG92] (see also [OR01]), who proved the following result on closed manifolds M of dimension 3: the fundamental group of M is free and nontrivial if and only if its LS category is 2. Furthermore, they conjectured that the fundamental group of every closed n-manifold, $n \geq 3$, of LS category 2 is necessarily free [GG92, Remark, p. 797]. Our interest in this natural problem was also stimulated in part by recent work on the comparison of the LS category and the systolic category [KR06, KR05, Ka07], which was inspired, in turn, by M. Gromov's systolic inequalities [Gr83, Gr96, Gr99, Gr07].

In the present text we prove this 1992 conjecture. Recall that all closed surfaces different from S^2 are of LS category 2.

1.1. **Theorem.** *A closed connected manifold of LS category* 2 *either is a surface, or has free fundamental group.*

1.2. **Corollary.** *Every manifold* $M^n, n \geq 3$, *with non-free fundamental group satisfies* $\operatorname{cat}_{\mathrm{LS}}(M) \geq 3$.

We found that there is no restriction on the fundamental group for closed manifolds of LS category 3. In particular we proved the following.

1.3. **Theorem.** *Given a finitely presented group* π *and non-negative integers* k, l, *there exists a closed manifold* M *such that* $\pi_1(M) = \pi$, *while* $\operatorname{cat}_{\mathrm{LS}} M = 3 + k$ *and* $\dim M = 5 + 2k + l$. *Furthermore, if* π *is not free, then* M *can be chosen* 4*-dimensional with* $\operatorname{cat}_{\mathrm{LS}} M = 3$.

Thus, there is no restriction on the fundamental group of manifolds of LS category 3 and higher.

The above results lead to the following questions:

1.4. **Question.** If a 4-dimensional CW-complex X has free fundamental group, then we have the bound $\operatorname{cat}_{\mathrm{LS}} X \leq 3$. Is the stronger bound $\operatorname{cat}_{\mathrm{LS}} X \leq 2$ necessarily satisfied?

We prove the inequality $\operatorname{cat}_{\mathrm{LS}} M \leq n - 2$ for connected n-manifolds with free fundamental group and $n > 4$, see Proposition 4.4. In [Str07],

J. Strom proved a stronger inequality $\operatorname{cat}_{\mathrm{LS}} X \leq \frac{2}{3} \dim X$ for an arbitrary CW-space X. Later, it was proved in [Dr07] that if the fundamental group is free, then the bound

$$\operatorname{cat}_{\mathrm{LS}} X \leq \frac{1}{2} \dim X + 1, \tag{1.1}$$

is satisfied by every CW-complex X.

The above Question 1.4 has an affirmative answer when M is a closed orientable manifold, in view of a theorem due to J. A. Hillman [Hi04] which states that a closed 4-dimensional manifold with free fundamental group has a CW-decomposition in which the three-skeleton has the homotopy type of a wedge of spheres.

1.5. **Question.** Is it true that $\operatorname{cat}_{\mathrm{LS}}(M \setminus \{\mathrm{pt}\}) = 1$ for any closed manifold M with $\operatorname{cat}_{\mathrm{LS}} M = 2$? This is proved in [GG92] for the case $\dim M = 3$. A direct proof would imply the main theorem trivially.

1.6. **Question.** Given integers m and n, describe the fundamental groups of closed manifolds M with $\dim M = n$ and $\operatorname{cat}_{\mathrm{LS}} M = m$.

Note that in the case $m = n$, the fundamental group of M is of cohomological dimension $\geq n$, see e.g. the Berstein–Švarc Theorem 5.4. Thus, we can ask when the converse holds.

1.7. **Question.** Given a finitely presented group π and an integer $n \geq 4$ such that $H^n(\pi) \neq 0$, when can one find a closed manifold M satisfying $\pi_1(M) = \pi$ and $\dim M = \operatorname{cat}_{\mathrm{LS}} M = n$? Note that Proposition 5.12 shows that such a manifold M does not always exist.

A related numerical invariant called the *systolic category* can be thought of as a Riemannian analogue of the LS category [Ka07]. In [DKR08] we apply Corollary 1.2 to prove that the systolic category of a 4-manifold is a lower bound for its LS category.

1.8. **Theorem.** *Every closed orientable 4-manifold M satisfies the inequality* $\operatorname{cat}_{\mathrm{sys}}(M) \leq \operatorname{cat}_{\mathrm{LS}}(M)$.

In particular, this inequality implies that if a 4-manifold M has a free fundamental group then $\operatorname{cat}_{\mathrm{sys}}(M) = \operatorname{cat}_{\mathrm{LS}}(M)$. In a related development in systolic topology, an intriguing model for BS^3 built out of BS^1 was used in [BKSW08] to prove that the symmetric metric of the quaternionic projective space, contrary to expectation, is *not* its systolically optimal metric.

The proof of the main theorem proceeds roughly as follows. If the group $\pi := \pi_1(M)$ is not free, then by a result of J. Stallings and R. Swan, the group π is of cohomological dimension at least 2. We

then show that π carries a suitable nontrivial 2-dimensional cohomology class u with twisted coefficients, and of category weight 2. Viewing M as a subspace of $K(\pi, 1)$ that contains the 2-skeleton $K(\pi, 1)^{(2)}$, and keeping in mind the fact that the 2-skeleton carries the fundamental group, we conclude that the restriction (pullback) of u to M is nonzero and also has category weight 2. By Poincaré duality with twisted coefficients, one can find a complementary $(n-2)$-dimensional cohomology class. By a category weight version of the cuplength argument, we therefore obtain a lower bound of 3 for $\operatorname{cat}_{\mathrm{LS}} M$.

In Section 2, we review the material on local coefficient systems, a twisted version of Poincaré duality, and 2-dimensional cohomology of non-free groups. In Section 3, we review the notion of category weight. In Section 4, we prove our main result, Theorem 1.1. In Section 5 we prove Theorem 1.3.

2. Cohomology with local coefficients

A *local coefficient system* $\mathcal{A}$ on a path connected CW-space X is a functor from the fundamental groupoid $\Gamma(X)$ of X, to the category of abelian groups. See [Ha02], [Wh78] for the definition and properties of local coefficient systems.

In other words, an abelian group $\mathcal{A}_x$ is assigned to each point $x \in X$, and for each path α joining x to y, an isomorphism $\alpha^* : \mathcal{A}_y \to \mathcal{A}_x$ is given. Furthermore, paths that are homotopic are required to yield the same isomorphism.

Let $\pi = \pi_1(X)$, and let $\mathbb{Z}[\pi]$ be the group ring of π. Note that all the groups A_x are isomorphic to a fixed group A. We will refer to A as a *stalk* of $\mathcal{A}$.

Given a map $f : Y \to X$ and a local coefficient system $\mathcal{A}$ on X, we define a local coefficient system on Y, denoted $f^*\mathcal{A}$, as follows. The map f yields a functor $\Gamma(f) : \Gamma(Y) \to \Gamma(X)$, and we define $f^*\mathcal{A}$ to be the functor $\mathcal{A} \circ \Gamma(f)$. Given a pair of coefficient systems $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is defined by setting $(\mathcal{A} \otimes \mathcal{B})_x = \mathcal{A}_x \otimes \mathcal{B}_x$.

2.1. **Example.** A useful example of a local coefficient system is given by the following construction. Given a fiber bundle $p : E \to X$ over X, set $F_x = p^{-1}(x)$. Then the family $\{H_k(F_x)\}$ can be regarded a local coefficient system, see [Wh78, Example 3, Ch. VI, §1]. An important special case is that of an n-manifold M and spherical tangent bundle $p : E \to M$ with fiber S^{n-1}, yielding a local coefficient system $\mathcal{O}$ with $\mathcal{O}_x = H_{n-1}(S^{n-1}_x) \cong \mathbb{Z}$. This local system is called the *orientation sheaf* of M.

2.2. **Remark.** There is a bijection between local coefficients on X and $\mathbb{Z}[\pi]$-modules [Sp66, Ch. 1, Exercises F]. If $\mathcal{A}$ is a local coefficient system with stalk A, then the natural action of the fundamental group on A turns A into a $\mathbb{Z}[\pi]$-module. Conversely, given a $\mathbb{Z}[\pi]$-module A, one can construct a local coefficient system $\mathcal{L}(A)$ such that induced $\mathbb{Z}[\pi]$-module structure on A coincides with the given one, cf. [Ha02].

We recall the definition of the (co)homology groups with local coefficients via modules [Ha02]:

$$H^k(X;\mathcal{A}) \cong H^k(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}), A), \delta) \tag{2.1}$$

and

$$H_k(X;\mathcal{A}) \cong H_k(A \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}), 1 \otimes \partial). \tag{2.2}$$

Here $(C_*(\widetilde{X}), \partial)$ is the chain complex of the universal cover $\widetilde{X}$ of X, A is the stalk of the local coefficient system $\mathcal{A}$, and δ is the coboundary operator. Note that in the tensor product we used the right $\mathbb{Z}[\pi]$ module structure on A defined via the standard rule $ag = g^{-1}a$, for $a \in A, g \in \pi$.

Recall that for CW-complexes X, there is a natural bijection between equivalence classes of local coefficient systems and locally constant sheaves on X. One can therefore define (co)homology with local coefficients as the corresponding sheaf cohomology [Bre97]. In particular, we refer to [Bre97] for the definition of the cup product

$$\cup : H^i(X;\mathcal{A}) \otimes H^j(X;\mathcal{B}) \to H^{i+j}(X;\mathcal{A}\otimes\mathcal{B})$$

and the cap product

$$\cap : H_i(X;\mathcal{A}) \otimes H^j(X;\mathcal{B}) \to H_{i-j}(X;\mathcal{A}\otimes\mathcal{B}).$$

A nice exposition of the cup and the cap products in a slightly different setting can be found in [Bro94]. In particular, we have the cap product

$$H_k(X;\mathcal{A}) \otimes H^k(X;\mathcal{B}) \to H_0(X;\mathcal{A}\otimes\mathcal{B}) \cong A \otimes_{\mathbb{Z}[\pi]} B.$$

2.3. **Proposition.** *Given an integer $k \geq 0$, there exists a local coefficient system $\mathcal{B}$ and a class $v \in H^k(X;\mathcal{B})$ such that, for every local coefficient system $\mathcal{A}$ and nonzero class $a \in H_k(X;\mathcal{A})$, we have $a \cap v \neq 0$.*

Proof. Throughout the proof $\otimes$ denotes $\otimes_{\mathbb{Z}[\pi]}$. We convert the stalk of $\mathcal{A}$ into a right $\mathbb{Z}[\pi]$-module A as above. Below we use the isomorphisms (2.1) and (2.2). Consider the chain $\mathbb{Z}[\pi]$-complex

$$\dots \longrightarrow C_{k+1}(\widetilde{X}) \xrightarrow{\partial_{k+1}} C_k(\widetilde{X}) \xrightarrow{\partial_k} C_{k-1}(\widetilde{X}) \longrightarrow \dots .$$

For the given k, we set $B := C_k(\widetilde{X})/\operatorname{Im}\partial_{k+1}$. Let $\mathcal{B}$ be the corresponding local system on X. Thus, we obtain the exact sequence of $\mathbb{Z}[\pi]$-modules

$$C_{k+1}(\widetilde{X}) \xrightarrow{\partial_{k+1}} C_k(\widetilde{X}) \xrightarrow{f} B \to 0.$$

Note that the epimorphism f can be regarded as a k-cocycle with values in $\mathcal{B}$, since $\delta f(x) = f\partial_{k+1}(x) = 0$. Let $v := [f] \in H^k(X;\mathcal{B})$ be the cohomology class of f. Now we prove that

$$a \cap [f] \neq 0.$$

Since the tensor product is right exact, we obtain the diagram

$$\begin{tikzcd} A \otimes C_{k+1}(\widetilde{X}) \arrow[r, "1\otimes\partial_{k+1}"] & A \otimes C_k(\widetilde{X}) \arrow[r, "1\otimes f"] & A \otimes B \arrow[r] \arrow[d, "g"'] & 0 \\ & & A \otimes C_{k-1}(\widetilde{X}) & \end{tikzcd}$$

where the row is exact. The composition

$$A \otimes C_k(\widetilde{X}) \xrightarrow{1\otimes f} A \otimes B \xrightarrow{g} A \otimes C_{k-1}(\widetilde{X})$$

coincides with $1 \otimes \partial_k$. We represent the class a by a cycle

$$z \in A \otimes C_k(\widetilde{X}).$$

Since $z \notin \operatorname{Im}(1 \otimes \partial_{k+1})$, we conclude that

$$(1 \otimes f)(z) \neq 0 \in A \otimes B = H_0(X; \mathcal{A} \otimes \mathcal{B}).$$

Thus, for the cohomology class v of f we have $a \cap v \neq 0$. □

Every closed connected n-manifold M satisfies $H_n(M;\mathcal{O}) \cong \mathbb{Z}$. A generator (one of two) of this group is called the *fundamental class* of M and is denoted by $[M]$.

One has the following generalization of the Poincaré duality isomorphism.

2.4. **Theorem** ([Bre97, Corollary 10.2]). *The homomorphism*

$$\Delta : H^i(M;\mathcal{A}) \to H_{n-i}(M;\mathcal{O}\otimes\mathcal{A}) \tag{2.3}$$

defined by setting $\Delta(a) = [M] \cap a$, *is an isomorphism.*

In fact, in [Bre97] there is the sheaf $\mathcal{O}^{-1}$ at the right, but for manifolds we have $\mathcal{O} = \mathcal{O}^{-1}$.

Given a group π and a $\mathbb{Z}[\pi]$-module A, we denote by $H^*(\pi; A)$ the cohomology of the group π with coefficients in A, see e.g. [Bro94]. Recall that $H^i(\pi; A) = H^i(K(\pi,1); \mathcal{L}(A))$, see Remark 2.2.

Let $\operatorname{cd}(\pi)$ denote the cohomological dimension of π over $\mathbb{Z}$, i.e. the largest m such that there exists an $\mathbb{Z}[\pi]$-module A with $H^m(\pi; A) \neq 0$.

2.5. **Theorem** ([Sta68, Swan69]). *If* $\operatorname{cd}\pi \le 1$ *then* π *is a free group.*

We will need the following known fact from the cohomology theory of groups.

2.6. **Lemma.** *If* π *be a group with* $\operatorname{cd}\pi = q \ge 2$. *Then* $H^2(\pi; A) \ne 0$ *for some* $\mathbb{Z}[\pi]$*-module* A.

Proof. We use the fact that cohomology of the group π with coefficients in an injective $\mathbb{Z}[\pi]$-module are trivial and the fact that every $\mathbb{Z}[\pi]$-module A' can be imbedded into an injective $\mathbb{Z}[\pi]$module J, [Bro94]. Let $0 \to A' \to J \to A'' \to 0$ be an exact sequence of $\mathbb{Z}[\pi]$-modules with J injective. Then by the coefficients long exact sequence $H^k(\pi; A') = H^{k-1}(\pi; A'')$ for $k > 1$. Since $H^q(\pi; B) \ne 0$ for some B, the proof can be completed by an obvious induction. □

3. Category weight and lower bounds for $\operatorname{cat}_{\mathrm{LS}}$

In this section, we review the notion of category weight and its relation to the Lusternik-Schnirelmann category.

3.1. **Definition** ([BG61, Fe53, Fo41]). Let $f : X \to Y$ be a map of (locally contractible) CW-spaces. The *Lusternik–Schnirelmann category of* f, denoted $\operatorname{cat}_{\mathrm{LS}}(f)$, is defined to be the minimal integer k such that there exists an open covering $\{U_0, \dots, U_k\}$ of X with the property that each of the restrictions $f|A_i \colon A_i \to Y$, $i = 0, 1, \dots, k$ is null-homotopic.

The *Lusternik–Schnirelmann category* $\operatorname{cat}_{\mathrm{LS}} X$ *of a space* X is defined as the category $\operatorname{cat}_{\mathrm{LS}}(1_X)$ of the identity map.

3.2. **Definition.** The *category weight* $\operatorname{wgt}(u)$ of a non-zero cohomology class $u \in H^*(X; \mathcal{A})$ is defined as follows:

$$\operatorname{wgt}(u) \ge k \iff \{\varphi^*(u) = 0 \text{ for every } \varphi\colon F \to X \text{ with } \operatorname{cat}_{\mathrm{LS}}(\varphi) < k\}.$$

3.3. **Remark.** E. Fadell and S. Husseini (see [FH92]) originally proposed the notion of category weight. In fact, they considered an invariant similar to the wgt of (3.2) (denoted in [FH92] by cwgt), but where the defining maps $\varphi\colon F \to X$ were required to be inclusions rather than general maps. As a consequence, cwgt is not a homotopy invariant, and thus a delicate quantity in homotopy calculations. Yu. Rudyak [Ru97, Ru99] and J. Strom [Str97] proposed a homotopy invariant version of category weight as defined in Definition 3.2.

3.4. **Proposition** ([Ru97, Str97]). *Category weight has the following properties.*

(1) $1 \le \operatorname{wgt}(u) \le \operatorname{cat}_{\mathrm{LS}}(X)$, *for all* $u \in \widetilde{H}^*(X; \mathcal{A}), u \ne 0$.

(2) *For every* $f\colon Y \to X$ *and* $u \in H^*(X;\mathcal{A})$ *with* $f^*(u) \neq 0$ *we have* $\operatorname{cat}_{\mathrm{LS}}(f) \geq \operatorname{wgt}(u)$ *and* $\operatorname{wgt}(f^*(u)) \geq \operatorname{wgt}(u)$.

(3) *For* $u \in H^*(X;\mathcal{A})$ *and* $v \in H^*(X;\mathcal{B})$ *we have*

$$\operatorname{wgt}(u \cup v) \geq \operatorname{wgt}(u) + \operatorname{wgt}(v).$$

(4) *For every* $u \in H^s(K(\pi,1);\mathcal{A})$, $u \neq 0$, *we have* $\operatorname{wgt}(u) \geq s$.

Proof. See [CLOT03, §2.7 and Proposition 8.22], the proofs in loc. cit. can be easily adapted to local coefficient systems. □

4. Manifolds of LS category 2

In this section we prove that the fundamental group of a closed connected manifold of LS category 2 is free.

4.1. **Theorem.** *Let* M *be a closed connected manifold of dimension at least* 3. *If the group* $\pi := \pi_1(M)$ *is not free, then* $\operatorname{cat}_{\mathrm{LS}} M \geq 3$.

Proof. By Theorem 2.5 and Lemma 2.6, there a local coefficient system $\mathcal{A}$ on $K(\pi,1)$ such that $H^2(K(\pi,1);\mathcal{A}) \neq 0$. Choose a non-zero element $u \in H^2(K(\pi,1);\mathcal{A})$. Let $f : M \to K(\pi,1)$ be the map that induces an isomorphism of fundamental groups, and let $i : K \to M$ be the inclusion of the 2-skeleton. (If M is not triangulable, we take i to be any map of a 2-polyhedron that induces an isomorphism of fundamental groups.) Then

$$(fi)^* : H^2(K(\pi,1);\mathcal{A}) \to H^2(K;(fi)^*\mathcal{A})$$

is a monomorphism. In particular, we have $f^*u \neq 0$ in $H^2(M;(f)^*\mathcal{A})$. Now consider the class

$$a = [M] \cap f^*u \in H_{n-2}(M;\mathcal{O}^{-1} \otimes f^*\mathcal{A}),$$

where $n = \dim M$. Then $a \neq 0$ by Theorem 2.4. Hence, by Proposition 2.3, there exists a class $v \in H^{n-2}(M;\mathcal{B})$ such that $a \cap v \neq 0$. We claim that $f^*u \cup v \neq 0$. Indeed, one has

$$[M] \cap (f^*u \cup v) = ([M] \cap f^*u) \cap v = a \cap v \neq 0.$$

Now, $\operatorname{wgt} f^*u \geq 2$ by Proposition 3.4, items (2) and (4). Furthermore, $\operatorname{wgt}(v) \geq 1$ by Proposition 3.4, item (1). We therefore obtain the lower bound $\operatorname{wgt}(f^*u \cup v) \geq 3$ by Proposition 3.4, item (3). Since $f^*u \cup v \neq 0$, we conclude that $\operatorname{cat}_{\mathrm{LS}} M \geq 3$ by Proposition 3.4, item (1). □

4.2. **Corollary.** *If* $M^n, n \geq 3$ *is a closed manifold with* $\operatorname{cat}_{\mathrm{LS}} M \leq 2$, *then* $\pi_1(M)$ *is a free group.*

4.3. **Remark.** An alternative approach to Theorem 4.1 would be using the Berstein-Švarc class $\mathfrak{b} \in H^1(\pi; I(\pi))$ where $I(\pi)$ is the augmentation ideal of π. If $\mathrm{cd}(\pi) \geq 2$ then $\mathfrak{b}^2 \neq 0$ by [DR07] (see also Theorem 5.4). In particular, $H^2(\pi; I(\pi) \otimes I(\pi)) \neq 0$, and we obtain an alternative proof of Lemma 2.6.

The following Proposition is a special case of [Dr07, Corollary 4.2]. Here we give a relatively simple geometric proof.

4.4. **Proposition.** *Let M be a closed connected n-dimensional PL manifold, $n > 4$, with free fundamental group. Then* $\mathrm{cat}_{\mathrm{LS}} M \leq n - 2$.

Proof. If X is a 2-dimensional (connected) CW-complex with free fundamental group then $\mathrm{cat}_{\mathrm{LS}} X \leq 1$, see e.g. [KRS06, Theorem 12.1]. Hence, if Y is a k-dimensional complex with free fundamental group then $\mathrm{cat}_{\mathrm{LS}} Y \leq k - 1$ for $k > 2$. Now, let K be a triangulation of M, and let L be its dual triangulation. Then $M \setminus L^{(l)}$ is homotopy equivalent to $K^{(k)}$ whenever $k + l + 1 = n$. Hence,

$$\mathrm{cat}_{\mathrm{LS}} M \leq \mathrm{cat}_{\mathrm{LS}} K^{(k)} + \mathrm{cat}_{\mathrm{LS}} L^{(l)} + 1.$$

Since $\pi_1(K)$ and $\pi_1(L)$ are free, we conclude that $\mathrm{cat}_{\mathrm{LS}} K^{(k)} \leq k-1$ and $\mathrm{cat}_{\mathrm{LS}} L^l \leq l-1$ for $k, l > 1$. Thus $\mathrm{cat}_{\mathrm{LS}} M \leq k-1+l-1+1 = n-2$. □

5. Manifolds of higher LS category

Gromov [Gr99, 4.40] called a polyhedron X *n-essential* if there is no map $f : X \to K(\pi, 1)^{(n-1)}$ to the $(n-1)$-dimensional skeleton of an Eilenberg-MacLane complex that induces an isomorphism of the fundamental groups. We extend his definition as follows.

5.1. **Definition.** A CW-space X is called *strictly k-essential*, $k > 1$ if for every CW-complex structure on X there is no map between the skeleta $f : X^{(k)} \to K(\pi, 1)^{(k-1)}$ that induces an isomorphism of the fundamental groups.

Clearly, a strictly n-essential space is Gromov n-essential, while the converse is false. Furthermore, an n-dimensional polyhedron is strictly n-essential if it is Gromov n-essential.

5.2. **Theorem.** *Let M be a closed strictly k-essential manifold. If its dimension satisfies* $\dim M \geq k + 1$, *then its LS category also satisfies* $\mathrm{cat}_{\mathrm{LS}} M \geq k + 1$.

Proof. We first consider the case $k = 2$. If $\mathrm{cat}_{\mathrm{LS}} M \leq 2$, then, by Theorem 4.1, $\pi_1(M)$ is free. Hence there is a map $f : M \to \vee S^1$ that induces an isomorphism of the fundamental groups, and M is not strictly 2-essential.

Now assume $k \geq 3$. Let $K = K(\pi_1(M), 1)$. Consider a map

$$f : M^{(k-1)} \to K^{(k-1)}$$

such that the restriction $f|_{M^{(2)}}$ is the identity homeomorphism of the 2-skeleta $M^{(2)}$ and $K^{(2)}$. We consider the problem of extension of f to M.

We claim that the first obstruction $o(f) \in H^k(M; E)$ (taken with coefficients in a local system E with the stalk $\pi_{k-1}(K^{(k-1)})$) to the extension is not equal to zero.

Indeed, if $o(f) = 0$, then there exists a map $\overline{f} : M^{(k)} \to K^{(k-1)}$ which coincides with f on the $(k-2)$-skeleton. The map

$$\overline{f}_* : \pi_1(M^{(k)}) \to \pi_1(K^{(k-1)})$$

can be viewed as an endomorphism of $\pi_1(M)$ that is identical on generators, and therefore $\overline{f}_*$ is an isomorphism. Hence M is not strictly k-essential.

Consider the commutative diagram

$$\begin{tikzcd}
M^{(k-1)} \arrow[r, "f"] \arrow[d, "i"'] & K^{(k-1)} \arrow[r, "\mathrm{id}"] \arrow[d, "j"'] & K^{(k-1)} \\
M \arrow[r, "\tilde{f}"] & K &
\end{tikzcd}$$

where i and j are the inclusions of the skeleta. Let α be the first obstruction to the extension of id to a map $K \to K^{(k-1)}$. By commutativity of the above diagram, we have $o(f) = \tilde{f}^*(\alpha)$. Now, asserting as in the proof of Theorem 4.1, we get that $\tilde{f}^*(\alpha) \cup v \neq 0$ for some v with $\dim v = \dim M - k$. Since $\dim M > k$, we conclude that $\dim v \geq 1$ and thus $\operatorname{cat}_{\mathrm{LS}} M \geq k+1$. $\square$

5.3. **Remark.** If a closed manifold M^n is n-essential then $\operatorname{cat}_{\mathrm{LS}} M = n$, see e.g. [KR06] and [Ka07, Theorem 12.5.2].

The following theorem for $n \geq 3$ was proven in [Ber76, Theorem A] and [Sva66, Theorem 20], see also [CLOT03, Proposition 2.51]. The case $n = 2$ was proved in [DR07].

5.4. **Theorem.** *If* $\dim X = \operatorname{cat}_{\mathrm{LS}} X = n$*, then* $u_X^n \neq 0$ *where* $u_X = j^*(\mathfrak{b}) \in H^1(X; I(\pi))$*,* $j : X \to K(\pi, 1)$ *induces an isomorphism of the fundamental groups, and* $\mathfrak{b} \in H^1(\pi, I(\pi))$ *is the Berstein-Švarc class. (For the case* $n = \infty$ *this means that* $u^k \neq 0$ *for all* k*.)*

5.5. **Proposition.** *For every non-free finitely presented group* π*, there exists a closed* 4*-dimensional manifold* M *with fundamental group* π *and* $\operatorname{cat}_{\mathrm{LS}} M = 3$.

Proof. Let K be a 2-skeleton of $K(\pi, 1)$. Take an embedding of K in $\mathbb{R}^5$ and let $M = \partial N$ be the boundary of the regular neighborhood N of this skeleton. Then there is a retraction $N \to K$, and, clearly, the map $f : M \subset N \to K$ induces an isomorphism of fundamental groups. Now, let $u_M \in H^1(M; I(\pi))$ be the class described in the Theorem 5.4. Then $u_M = f^* u_K$, and hence $u_M^4 = 0$. Therefore $\operatorname{cat}_{\mathrm{LS}} M < 4$ by Theorem 5.4, and thus $\operatorname{cat}_{\mathrm{LS}} M = 3$. □

Let M_f be the mapping cylinder of $f : X \to Y$. We use the notation $\pi_*(f) = \pi_*(M_f, X)$. Then $\pi_i(f) = 0$ for $i \leq n$ amounts to saying that it induces isomorphisms $f_* : \pi_i(X_1) \to \pi_i(Y_1)$ for $i \leq n$ and an epimorphism in dimension $n+1$. Similar notation $H_*(f) = H_*(M_f, X)$ we use for homology.

5.6. **Lemma.** *Let $f_j : X_j \to Y_j$ be a family of maps of CW-spaces such that $H_i(f_j) = 0$ for $i \leq n_j$. Then $H_i(f_1 \wedge \cdots \wedge f_s) = 0$ for $i \leq \min\{n_j\}$.*

Proof. Note that

$$M(f_1 \wedge \cdots \wedge f_s) \cong Y_1 \wedge \cdots \wedge Y_s \cong M(f_1) \wedge \cdots \wedge M(f_s).$$

Now, by using the Künneth formula and considering the homology exact sequence of the pair $(M(f_1) \wedge \cdots \wedge M(f_s), X_1 \wedge \cdots \wedge X_s)$, we obtain the result. □

5.7. **Proposition.** *Let $f_j : X_j \to Y_j$, $3 \leq j \leq s$ be a family of maps of CW-spaces such that $\pi_i(f_j) = 0$ for $i \leq n_j$. Then the joins satisfy*

$$\pi_k(f_1 * f_2 * \cdots * f_s) = 0$$

for $k \leq \min\{n_j\} + s - 1$.

Proof. By the version of the Relative Hurewicz Theorem for non-simply connected X_j [Ha02, Theorem 4.37], we obtain $H_i(f_j) = 0$ for $i \leq n_j$. By Lemma 5.6 we obtain that $H_k(f_1 \wedge \cdots \wedge f_s) = 0$ for $k \leq \min\{n_j\}$. Since the join $A_1 * \cdots * A_s$ is homotopy equivalent to the iterated suspension $\Sigma^{s-1}(A_1 \wedge \cdots \wedge A_s)$ over the smash product, we conclude that $H_k(f_1 * \cdots * f_s) = 0$ for $k \leq \min\{n_j\} + s - 1$. Since $X_1 * \cdots * X_s$ is simply connected for $s \geq 3$, by the standard Relative Hurewicz Theorem we obtain that $\pi_k(f_1 * \cdots * f_s) = 0$ for $k \leq \min\{n_j\} + s - 1$. □

Given two maps $f : Y_1 \to X$ and $g : Y_2 \to X$, we set

$$Z = \{(y_1, y_2, t) \in Y_1 * Y_2 \mid f(y_1) = g(y_2)\}$$

and define the *fiberwise join*, or *join over X* of f and g as the map

$$f *_X g : Z \to X, \quad (f *_X g)(y_1, y_2, t) = f(y_1)$$

Let $p_0^X : PX \to X$ be the Serre path fibration. This means that PX is the space of paths on X that start at the base point of the pointed space X, and $p_0(\alpha) = \alpha(1)$. We denote by $p_n^X; G_n(X) \to X$ the n-fold fiberwise join of p_0.

The proof of the following theorem can be found in [CLOT03].

5.8. **Theorem** (Ganea, Švarc)**.** *For a CW-space X, $\operatorname{cat}_{\mathrm{LS}}(X) \le n$ if and only if there exists a section of $p_n : G_n(X) \to X$.*

5.9. **Proposition.** *The connected sum $S^k \times S^l \# \cdots \# S^k \times S^l$ is a space of LS-category 2.*

Proof. This can be deduced from a general result of K. Hardy [H73] because the connected sum of two manifolds can be regarded as the double mapping cylinder. Alternatively, one can note that, after removing a point, the manifold on hand is homotopy equivalent to the wedge of spheres. □

5.10. **Theorem.** *For every finitely presented group π and $n \ge 5$, there is a closed n-manifold M of LS-category 3 with $\pi_1(M) = \pi$.*

Proof. If the group π is the free group of rank s, we let M' be the k-fold connected sum $S^1 \times S^2 \# \cdots \# S^1 \times S^2$. Then M' is a closed 3-manifold of LS category 2 with $\pi_1(M') = F_s$. Then the product manifold $M = M' \times S^{n-3}$ has cuplength 3 and is therefore the desired manifold.

Now assume that the group π is not free. We fix a presentation of π with s generators and r relators. Let M' be the k-fold connected sum $S^1 \times S^{n-1} \# \cdots \# S^1 \times S^{n-1}$. Then M' is a closed n-manifold of the category 2 with $\pi_1(M') = F_s$. For every relator w we fix a nicely imbedded circle $S^1_w \subset M'$ such that $S^{-1}_w \cap S^{-1}_v = \emptyset$ for $w \ne v$. Then we perform the surgery on these circles to obtain a manifold M. Clearly, $\pi_1(M) = \pi$. We show that $\operatorname{cat}_{\mathrm{LS}}(M) \le 3$, and so $\operatorname{cat}_{\mathrm{LS}} M = 3$ by Theorem 4.1.

As usual, the surgery process yields an $(n+1)$-manifold X with $\partial X = M \sqcup M'$. Here X is the space obtained from $M' \times I$ by attaching handles $D^2 \times D^{n-1}$ of index 2 to $M' \times 1$ along the above circles. We note that $\operatorname{cat}_{\mathrm{LS}}(X) \le 3$.

On the other hand, by duality, X can be obtained from $M \times I$ by attaching handles of index $n-1$ to the boundary component of $M \times I$. In particular, the inclusion $f : M \to X$ induces an isomorphism of the homotopy groups of dimension $\le n-3$ and an epimorphism in dimension $n-2$. Hence the map

$$\Omega f : \Omega M \to \Omega X$$

induces isomorphisms in dimensions $\leq n-4$ and an epimorphism in dimension $n-3$. Thus, $\pi_i(\Omega f) = 0$ for $i \leq n-3$.

In order to prove the bound $\operatorname{cat}_{\mathrm{LS}} M \leq 3$, it suffices to show that the Ganea-Švarc fibration $p_3 : G_3(M) \to M$ has a section. Consider the commutative diagram

$$\begin{array}{ccccc}
G_3M & \xrightarrow{q} & Z & \xrightarrow{f'} & G_3(X) \\
{\scriptstyle p^3_M}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p^X_3} \\
M & = & M & \xrightarrow{f} & X
\end{array}$$

where the right-hand square is the pull-back diagram and $f'q = G_3(f)$. Note that q is uniquely determined. Since $\operatorname{cat}_{\mathrm{LS}}(X) \leq 3$, by Theorem 5.8 there is a section $s : X \to G_3(X)$. It defines a section $s' : M \to Z$ of p'. It suffices to show that the map $s' : M \to Z$ admits a homotopy lifting $h : M \to G_3M$ with respect to q, i.e. the map h with $qh \cong s'$. Indeed, we have

$$p^3_M h = p'qh \cong p's' = 1_M$$

and so h is a homotopy section of p^M_3. Since the latter is a Serre fibration, the homotopy lifting property yields an actual section.

Let F_1 and F_2 be the fibers of fibrations p^M_3 and p', respectively. Consider the commutative diagram generated by the homotopy exact sequences of the Serre fibrations p^M_3 and p':

$$\begin{array}{ccccccccc}
\pi_i(F_1) & \longrightarrow & \pi_i(G_3(M)) & \xrightarrow{(p^M_3)_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(F_1) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \phi_*} & & \\
\pi_i(F_2) & \longrightarrow & \pi_i(Z) & \xrightarrow{(p')_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(F_2) & \longrightarrow & \cdots .
\end{array}$$

Note that we have

$$\phi = \Omega(f) * \Omega(f) * \Omega(f) * \Omega(f).$$

By Proposition 5.7 and since $\pi_i(\Omega f) = 0$ for $i \leq n-3$, we conclude that $\pi_i(\phi) = 0$ for $i \leq n-3+3 = n$. Hence ϕ induces an isomorphism of the homotopy groups of dimensions $\leq n-1$ and an epimorphism in dimension n. By the Five Lemma we obtain that q_* is an isomorphism in dimensions $\leq n-1$ and an epimorphism in dimension n. Hence the homotopy fiber of q is $(n-1)$-connected. Since $\dim M = n$, the map s' admits a homotopy lifting $h : M \to G_3(M)$. □

5.11. **Corollary.** *Given a finitely presented group π and non-negative integer numbers k, l there exists a closed manifold M such that $\pi_1(M) = \pi$, while* $\operatorname{cat}_{\mathrm{LS}} M = 3 + k$ *and* $\dim M = 5 + 2k + l$.

Proof. By Theorem 5.10, there exists a manifold N such that $\pi_1(M) = \pi$, $\operatorname{cat}_{\mathrm{LS}} M = 3$ and $\dim M = 5+l$. Moreover, this manifold N possesses a detecting element, i.e. a cohomology class whose category weight is equal to $\operatorname{cat}_{\mathrm{LS}} N = 3$. For π free this follows since the cuplength of N is equal to 3, for other groups we have the detecting element $f^*u \cup v$ constructed in the proof of Theorem 4.1. If a space X possesses a detecting element then, for every $k > 0$, we have $\operatorname{cat}_{\mathrm{LS}}(X \times S^k) = \operatorname{cat}_{\mathrm{LS}} X + 1$ and $X \times S$ possesses a detecting element, [Ru99]. Now, the manifold $M := N \times (S^2)^k$ is the desired manifold. □

Generally, we have a question about relations between the category, the dimension, and the fundamental group of a closed manifold. The following proposition shows that the situation quite intricate.

5.12. **Proposition.** *Let p be an odd prime. Then there exists a closed $(2n+1)$-manifold with $\operatorname{cat}_{\mathrm{LS}} M = \dim M$ and $\pi_1(M) = \mathbb{Z}_p$, but there are no closed $2n$-manifolds with $\operatorname{cat}_{\mathrm{LS}} M = \dim M$ and $\pi_1(M) = \mathbb{Z}_p$.*

Proof. An example of $(2n+1)$-manifold is the quotient space $S^{2n+1}/\mathbb{Z}_p$ with respect to a free $\mathbb{Z}_p$-action on S^{2n+1}. Now, given a $2n$-manifold with $\pi_1(M) = \mathbb{Z}_p$, consider a map $f : M \to K(\mathbb{Z}_p, 1)$ that induces an isomorphism of fundamental groups. Since $H_{2n}(K(\mathbb{Z}_p, 1)) = 0$, it follows from the obstruction theory and Poincaré duality that f can be deformed into the $(2n-1)$-skeleton of $K(\mathbb{Z}_p, 1)$, cf. [Ba93, Section 8]. Hence, M is not $2n$-essential, and thus $\operatorname{cat}_{\mathrm{LS}} M < 2n$ [KR06]. □

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